# A semantic account of strong normalization in Linear Logic

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**Abstract**

We prove that given two cut free nets of linear logic, by means of their relational interpretations one can: 1) first determine whether or not the net obtained by cutting the two nets is strongly normalizable 2) then (in case it is strongly normalizable) compute the maximal length of the reduction sequences starting from that net.

## 1 Introduction

This paper follows the approach to the semantics of bounded time complexity consisting in measuring by semantic means the execution of any program, regardless to its computational complexity. The aim is to compare different computational behaviors and to learn afterwards something on the very nature of bounded time complexity. Following this approach and inspired by [7], in [4, 5] one of the authors of the present paper could compute the execution time of an untyped $\lambda$-term from its interpretation in the Kleisli category of the comonad associated with the finite multisets functor on the category of sets and relations. Such an interpretation is the same as the interpretation of the net translating the $\lambda$-term in the multiset based relational model of linear logic. The execution time is measured here in terms of elementary steps of the so-called Krivine machine. Also, [4, 5] give a precise relation between an intersection types system introduced by [2] and experiments in the multiset based relational model. Experiments are a tool introduced by Girard in [8] allowing to compute the interpretation of proofs pointwise. An experiment corresponds to a type derivation and the result of an experiment corresponds to a type. This same approach was applied in [6] to Multiplicative Exponential Linear Logic (MELL) to show how it is possible to compute the number of steps of cut elimination by semantic means (notice that the measure being now the number of cut elimination steps, here is a first difference with [4, 5] where Krivine's machine was used

to measure execution time). The results of [6] are presented in the framework of proof-nets, that we call nets in this paper: if $\pi'$ is a net obtained by applying some steps of cut elimination to $\pi$, the main property of any model is that the interpretation $[\![\pi]\!]$ of $\pi$ is the same as the interpretation $[\![\pi']\!]$ of $\pi'$, so that from $[\![\pi]\!]$ it is clearly impossible to determine the number of steps leading from $\pi$ to $\pi'$. Nevertheless, in [6] it is shown that if $\pi_1$ and $\pi_2$ are two cut free nets connected by means of a cut-link, one can answer the two following questions by only referring to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the relational model:

- is it the case that the net obtained by cutting $\pi_1$ and $\pi_2$ is weakly normalizable?
- if the answer to the previous question is positive, what is the number of cut reduction steps leading from the net with cut to a cut free one?

In the present paper, still by only referring to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the relational model, we answer the two following variants of the previous questions:

1. is it the case that the net obtained by cutting $\pi_1$ and $\pi_2$ is strongly normalizable?
2. if the answer to the previous question is positive, what is the maximal length (i.e. the number of cut reduction steps) of the reduction sequences starting from the net obtained by cutting $\pi_1$ and $\pi_2$?

It is certainly worth comparing our result with the recent paper [1], which gives a semantic bound of the number of $\beta$-reductions of a given $\lambda$-term.

The first question makes sense only in an untyped framework (in the typed case, cut elimination is strongly normalizing, see [8] and [9]), and we thus introduce in Section 2 nets and their reduction in an untyped framework. This section essentially comes from [6], so as several notions in the sequel, which are only recalled here.

In Section 3, we introduce the standard notion of experiment (called $[\![]\!]$-experiment in this paper) leading to the usual interpretation (called $[\![]\!]$-interpretation in this paper) of a net in the category of sets and relation (the multiset based relational model of linear logic). In the same Definition 10, we introduce $(\!|\!)$-experiments, leading to the $(\!|\!)$-interpretation of nets: the main difference between $[\![]\!]$-experiments and $(\!|\!)$-experiments is the behavior w.r.t. weakening links. And indeed, the main difference between weak and strong normalization lies in the fact that to study the latter property we cannot "forget pieces of proofs" (and this is actually what the usual $[\![]\!]$-interpretation does by assigning the empty multiset as label to the conclusion of weakening links). The newly defined $(\!|\!)$-interpretation *does not* yield a model of linear logic: it is invariant only w.r.t. *non erasing* reduction steps (Proposition 12).

In Section 4 we start by a crucial remark: in case $\pi$ is a cut free net, its $(\!|\!)$-interpretation $(\!|\pi|\!)$ can be computed from its "good old" $[\![]\!]$-interpretation $[\![\pi]\!]$ (Proposition 19). This implies that to answer questions 1 and 2 by only referring

to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the "good old" relational model of linear logic, we are allowed to use the newly defined $(\!|\,|\!)$-interpretations $(\!|\pi_1|\!)$ and $(\!|\pi_2|\!)$. We then accurately adapt the notion of size of an $[\![\,]\!]$-experiment of the relational model to $(\!|\,|\!)$-experiments, in order to obtain a variant of the "Key Lemma" (actually Lemmata 17 and 20 of [6]): Lemma 22, measuring the difference between the size of experiments of a net and one of its one step reducts. Corollary 24 answers question 1. Finally, we use a variant of the (rather standard) "postponement" of erasing reduction steps (Proposition 7, whose detailed proof can be found in the appendix), and again by accurately adapting the notion of size of a result of a $[\![\,]\!]$-experiment to $(\!|\,|\!)$-experiments (Definition 26), we state and prove our main result (Theorem 34), that answers question 2.

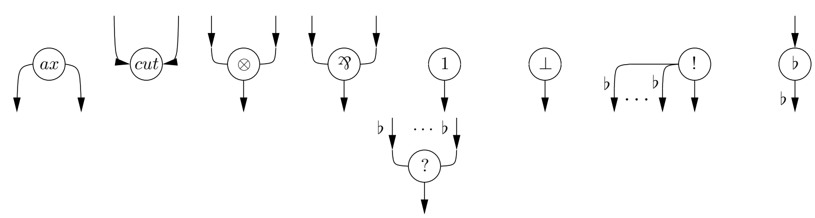
### Notations

For a set $X$, $\mathcal{P}_f(X)$ denotes the set of the finite subsets of $X$ and $\mathcal{M}_{\text{fin}}(X)$ denotes the set of finite multisets of elements of $X$.

## 2 Nets

We introduce nets and their cut-elimination in an untyped framework; we then prove some useful rewriting properties that will be used in the next sections. Most of the basic definitions come from [6].

**Definition 1** (Ground-structure)**.** *A* ground-structure*, or* g-structure *for short, is a finite (possibly empty) labelled directed acyclic graph whose nodes (also called links) are defined together with an arity and a coarity, i.e. a given number of incident edges called the* premises *of the node and a given number of emergent edges called the* conclusions *of the node. The valid nodes are:*




*An edge can have or not a ♭ label: an edge with no label (resp. with a ♭ label) is called* logical *(resp.* structural*). The ♭-nodes have a logical premise and a structural conclusion, the ?-nodes have $k \geq 0$ structural premises and one logical conclusion, the !-nodes have no premise, exactly one logical conclusion, also called* main *conclusion of the node, and $k \geq 0$ structural conclusions, called* auxiliary *conclusions of the node. Premises and conclusions of the nodes ax, cut, ⊗, ⅋, 1, ⊥ are logical edges. We allow edges with a source but no target, they are called* conclusions *of the g-structure; we consider that a g-structure is given with an order $(c_1, \ldots, c_n)$ of its conclusions.*
*We denote by $!(\alpha)$ the set of $!$-links of a g-structure $\alpha$.*

In the sequel we will not write explicitly the orientation of the edges. In order to give more concise pictures, when not misleading, we may represent an arbitrary number of $\flat$-edges (possibly zero) as a $\flat$-edge with a diagonal stroke drawn across. In the same spirit, a ?-link with a diagonal stroke drawn across its conclusion represents an arbitrary number of ?-links, possibly zero.

**Definition 2** (Untyped $\flat$-structure, untyped nets)**.** *An* untyped $\flat$-structure*, or simply* $\flat$-structure*,* $\pi$ of depth 0 *is a g-structure without* !*-nodes; in this case, we set* $ground(\pi) = \pi$*. An* untyped $\flat$-structure $\pi$ of depth $d+1$ *is a g-structure* $\alpha$*, denoted by* $ground(\pi)$*, with a function that assigns to every* !*-link* $o$ *of* $\alpha$ *with* $n_o + 1$ *conclusions a* $\flat$*-structure* $\pi^o$ *of depth at most* $d$*, called the* box of $o$*, with* $n_o$ *structural conclusions, also called* auxiliary conclusions *of* $\pi^o$*, and exactly one logical conclusion, called the* main conclusion *of* $\pi^o$*, and a bijection from the set of the* $n_o$ *structural conclusions of the link* $o$ *to the set of the* $n_o$ *structural conclusions of the* $\flat$*-structure* $\pi^o$*. Moreover* $\alpha$ *has at least one* !*-link with a box of depth* $d$*.*
*We say that* $ground(\pi)$ *is the* g-structure of depth 0 of $\pi$*; a* g-structure of depth $d+1$ in $\pi$ *is a g-structure of depth* $d$ *of the box associated by* $\pi$ *with a* !*-node of* $ground(\pi)$*. A* link $l$ of depth $d$ of $\pi$ *is a link of a g-structure of depth* $d$ *of* $\pi$*; we denote by* $depth(l)$ *the depth* $d$ *of* $l$*. We refer more generally to a link/g-structure of* $\pi$ *meaning a link/g-structure of some depth of* $\pi$*.*

*A* switching *of a g-structure* $\alpha$ *is an undirected subgraph of* $\alpha$ *obtained by forgetting the orientation of* $\alpha$*'s edges, by deleting one of the two premises of each* $⅋$*-node, and for every* ?*-node* $l$ *with* $n \geq 1$ *premises, by erasing all but one premises of* $l$*.*
*An* untyped $\flat$-net*,* $\flat$-net *for short, is a* $\flat$*-structure* $\pi$ *s.t. every switching of every g-structure of* $\pi$ *is an acyclic graph. An* untyped net*,* net *for short, is a* $\flat$*-net with no structural conclusion.*

Notice that with every structural edge $b$ of a net is associated exactly one $\flat$-node (above it) and one ?-node (below it): we will refer to these nodes as *the* $\flat$*-node/?-node associated with* $b$. Observe that the $\flat$-node and the ?-node associated with a given edge might have a different depth.

**Definition 3** (Size of nets)**.** *The* size $s(\alpha)$ of a g-structure $\alpha$ *is the number of logical edges of* $\alpha$*. The* size $s(\pi)$ of a $\flat$-structure $\pi$ *is defined by induction on the depth of* $\pi$*, as follows:* $s(\pi) = s(ground(\pi)) + \sum_{o \in !(ground(\pi))} s(\pi^o)$.

Since we are in an untyped framework, nets may contain "pathological" cuts which are not reducible. They are called *clashes* and their presence is in contrast with what happens in $\lambda$-calculus, where the simpler grammar of terms avoids clashes also in an untyped framework.

**Definition 4** (Clash)**.** *The two edges premises of a cut-link are* dual *when:*

- *they are conclusions of resp. a* $\otimes$*-node and of a* $⅋$*-node, or*
- *they are conclusions of resp. a* $1$*-node and of a* $\perp$*-node, or*

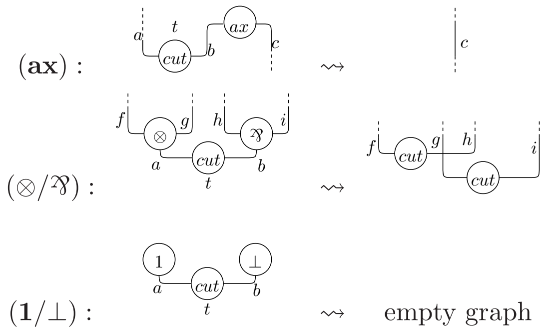



Figure 1: Cut elimination for nets (multiplicatives).

- *they are conclusions of resp. a* $!$*-node and of a* $?$*-node.*

*A cut-link is a* clash*, when the premises of the cut-node are not dual edges and none of the two is the conclusion of an ax-link.*

**Definition 5** (Cut elimination, Figures 1 and 2)**.** *The cut elimination is defined as in [3]. To eliminate a cut $t$ in a net $\pi$ means in general to transform $\pi$ into a net $t(\pi)$. We will also refer to $t(\pi)$ as a one step reduct of $\pi$, and to the transformations associated with the different types of cut-link as the* reduction steps*. When one of the two premises of $t$ is a $?$-link with no premises and the other one is a $!$-link, we say that $t$ is* erasing *and the reduction step is an erasing step. We write $\pi \rightsquigarrow \pi'$, when $\pi'$ is the result of one reduction step and $\pi \rightsquigarrow_e \pi'$ (resp. $\pi \rightsquigarrow_{\neg e} \pi'$) in case the reduction step is (resp. is not) erasing.*

*A cut-link $t$ of $\pi$ is* stratified non-erasing*, when for every non erasing cut (except clashes) $t'$ of $\pi$ we have $depth(t) \leq depth(t')$. A stratified non-erasing reduction step is a step reducing a stratified non-erasing cut; we write $\pi \rightsquigarrow_{(\neg e)_s} \pi'$ when $\pi'$ is the result of one stratified non-erasing reduction step.*
*A cut-link $t$ of $\pi$ is* antistratified erasing*, when every cut-link of $\pi$ is erasing and for every cut-link $t'$ of $\pi$ we have $depth(t') \leq depth(t)$. An antistratified erasing reduction step is a step reducing an antistratified erasing cut; we write $\pi \rightsquigarrow_{e_{as}} \pi'$ when $\pi'$ is the result of one antistratified erasing reduction step.*

*The reflexive and transitive closure of the rewriting rules previously defined is denoted by adding a $*$: for example $\rightsquigarrow^*_{(\neg e)_s}$ is the reflexive and transitive closure of $\rightsquigarrow_{(\neg e)_s}$. A net $\pi$ is* normalizable *if there exists a cut free net $\pi_0$ such that $\pi \rightsquigarrow^* \pi_0$.*
*A* reduction sequence *$R$ from $\pi$ to $\pi'$ is a sequence (possibly empty in case $\pi = \pi'$) of reduction steps $\pi \rightsquigarrow \pi_1 \rightsquigarrow \ldots \rightsquigarrow \pi_n = \pi'$. The integer $n$ is the* length *of the reduction sequence. A reduction sequence $R$ is a* stratified non-erasing reduction *(resp. an* antistratified erasing reduction*) when every step of $R$ is stratified non-erasing (resp. antistratified erasing). A net is $\neg$e-normal when it contains only erasing cut-links.*

(!/?) :

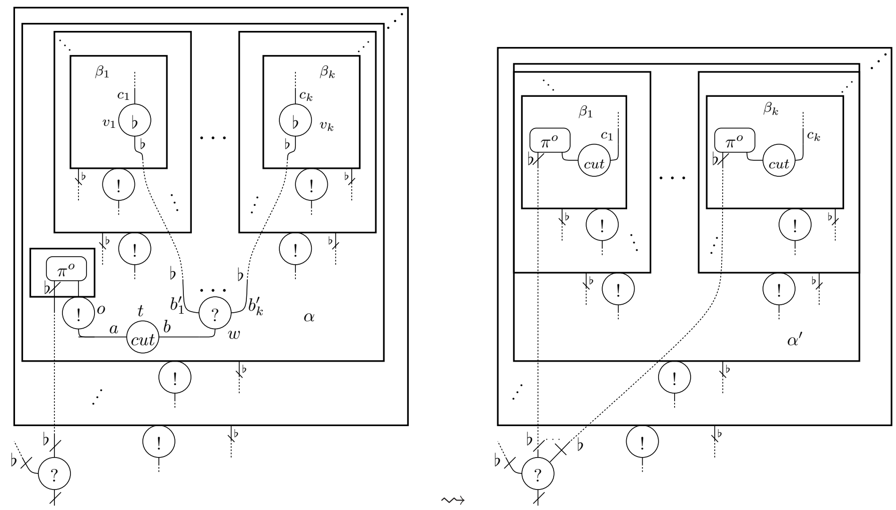



Figure 2: Cut elimination for nets. In the (!/?) case what happens is that the !-link $o$ dispatches $k$ copies of $\pi^o$ ($k \geq 0$ being the arity of the ?-node $w$ premise of the cut) inside the !-boxes (if any) containing the $\flat$-nodes associated with the premises of $w$; notice also that the reduction duplicates $k$ times the premises of ?-nodes which are associated with the auxiliary conclusions of $o$.

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*We denote by* $\boldsymbol{SN}$ *(resp.* $\boldsymbol{SN}^{\neg e}$*,* $\boldsymbol{SN}^{(\neg e)_s}$*) the set of nets* $\pi$ *such that every reduction sequence (resp.* $\neg$*e-reduction sequence,* $(\neg e)_s$*-reduction sequence) from* $\pi$ *is finite and none of the reducts (resp.* $\neg$*e-reducts,* $(\neg e)_s$*-reducts) of* $\pi$ *contains a clash. The nets of* $\boldsymbol{SN}$ *are also called* strongly normalizable*.*

*For any net* $\pi$*, we set*

- $strong_{\neg e}(\pi) = \begin{cases} \max\{length(R); R \text{ is a } \neg e\text{-reduction sequence from } \pi\} & \text{if } \pi \in \boldsymbol{SN}^{\neg e}; \\ \infty & \text{otherwise;} \end{cases}$
- *and* $strong(\pi) = \begin{cases} \max\{length(R); R \text{ is a reduction sequence from } \pi\} & \text{if } \pi \in \boldsymbol{SN}; \\ \infty & \text{otherwise.} \end{cases}$

In order to measure by semantic means the exact length of the longest reduction sequence(s) starting from a given strongly normalizable net (Theorem 34), we show that there always exists such a sequence consisting first of non erasing stratified steps and then of erasing antistratified steps (Proposition 7).

**Proposition 6.** *We have* $\boldsymbol{SN} = \boldsymbol{SN}^{(\neg e)_s}$*.*

**Proposition 7.** *For any* $\pi_0 \in \boldsymbol{SN}$*, there exist a* $(\neg e)_s$*-reduction sequence* $R_1 : \pi_0 \rightsquigarrow^*_{(\neg e)_s} \pi$ *with* $\pi$ $\neg$*e-normal and an antistratified e-reduction sequence* $R_2$ *from* $\pi$ *such that* $strong(\pi_0) = length(R_1) + length(R_2)$*.*

When $\pi$ (resp. $\pi'$) is a net having $c$ (resp. $c'$) among its conclusions, we denote in the sequel by $(\pi|\pi')_{c,c'}$ the net obtained by connecting $\pi$ and $\pi'$ by means of a *cut*-link with premises $c$ and $c'$.

**Corollary 8.** *Let $\pi$ (resp. $\pi'$) be a net having c (resp. c') among its conclusions, and assume that $(\pi|\pi')_{c,c'}$ is strongly normalizable.*

*There exists $R_1 : (\pi|\pi')_{c,c'} \rightsquigarrow^*_{(\neg e)_s} \pi_1$ and $R_2 : \pi_1 \rightsquigarrow_e{}^* \pi_2$ antistratified such that*

- *$\pi_1$ is $\neg e$-normal;*
- *$\pi_2$ is cut-free;*
- *$strong((\pi|\pi')_{c,c'}) = length(R_1) + length(R_2)$.*

# 3 Experiments

Let us fix a set $A$ of "atoms", such that $A$ does not contain any pair nor any multiset. We also require that $* \notin A$: these conditions on $A$ ensure that following Definition 9 we obtain an object $D$ that satisfies the equation $D = A \oplus A^\perp \oplus 1 \oplus \perp \oplus (D \otimes D) \oplus (D \parr D) \oplus !D \oplus ?D$, where the constructs have the usual interpretations: $A^\perp = A$, $\otimes$ and $\parr$ are the cartesian product of sets, 1 and $\perp$ are the singleton $\{*\}$, ! and ? are the finite multisets functor, and $\oplus$ is a disjoint union[1].

**Definition 9.** *We define $D_n$ by induction on $n$:*

- $D_0 := \{+, -\} \times (A \cup \{*\})$
- $D_{n+1} := D_0 \cup (\{+, -\} \times D_n \times D_n) \cup (\{+, -\} \times \mathcal{M}_{fin}(D_n))$

*We set $D := \bigcup_{n \in N} D_n$, and we call the depth of an element $x \in D$ (and we denote by $depth(x)$) the least $n$ such that $x \in D^n$.*

*When $(+, [])$ does not appear in $x \in D$, we say that $x$ is* exhaustive. *We denote by $X^{ex}$ the set of the exhaustive elements of any given subset $X$ of $D$.*

Now, we show how to compute the interpretation of an untyped net directly, without passing through a sequent calculus. This is done by adapting the notion of experiment to our untyped framework. For a net $\pi$ with $n$ conclusions, we define the *interpretation of* $\pi$, denoted by $[\![\pi]\!]$, as a subset of $\parr_{i=1}^n D$, that can be seen as a morphism from 1 to $\parr_{i=1}^n D$. We compute $[\![\pi]\!]$ by means of the $[\![]\!]$-*experiments of* $\pi$, a notion introduced by Girard in [8] and central in this paper. We introduce also a variant of this notion, the $(\!|\,|\!)$-*experiments of* $\pi$ that allow to compute $(\!|\pi|\!)$. We define, by induction on the depth of $\pi$, what are $[\![]\!]$-experiments and $(\!|\,|\!)$-experiments of $\pi$:

**Definition 10** (Experiment)**.** *An* $[\![]\!]$-experiment $e$ of a $\flat$-net $\pi$ *(resp. an* $(\!|\,|\!)$-experiment $e$ of a $\flat$-net $\pi$*), denoted by $e :_{[\![]\!]} \pi$ (resp. $e :_{(\!|\,|\!)} \pi$), is a function which*

[1] The previously mentioned conditions guarantee that the following definition of $D$ gives rise indeed to a *disjoint* union.

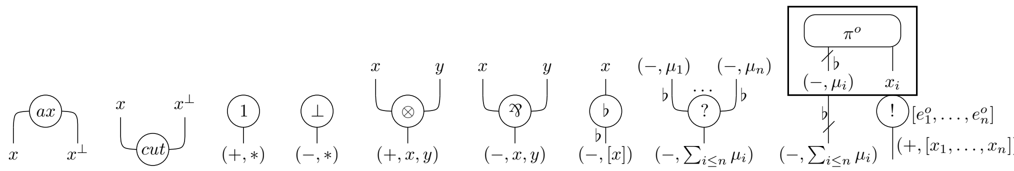


Figure 3: $\llbracket\rrbracket$-experiments of ♭-nets, with $x$, $y$, $x_i \in D$ and $\mu_i \in \mathcal{M}_{\mathrm{fin}}(D)$.

*associates with every !-link $o$ of ground($\pi$) a multiset $[e^o_1, ..., e^o_k]$ with $k \geq 0$ (resp. $k > 0$) of $\llbracket\rrbracket$-experiments (resp. $\llparenthesis\rrparenthesis$-experiments) of $\pi^o$, and with every edge $a$ of ground($\pi$) an element of $D$. In the cases of ax-links, cut-links, 1-links, $\perp$-link, $\otimes$-links, ⅋-links, ♭-links, !-links and ?-links with $n \geq 1$ premises, the standard conditions of Figure 3 hold both for $\llbracket\rrbracket$-experiments and $\llparenthesis\rrparenthesis$-experiments (see [6])[2]. In the case of a ?-link with no premise and the edge $c$ as conclusion, we require that:*

- *$e(c) = (-, [])$, for an $\llbracket\rrbracket$-experiment $e$*
- *$e(c) = (-, [])$ or $e(c) = (-, [\alpha])$ with $\alpha \in D$, for an $\llparenthesis\rrparenthesis$-experiment $e$.*

*When $e$ is an $\llbracket\rrbracket$-experiment (resp. an $\llparenthesis\rrparenthesis$-experiment), we set*[3]*:*

$$\begin{aligned}\mathcal{W}(e) \quad &= \sum_{\substack{c \text{ is the conclusion of a ?-link with no premise} \\ e(c)=(-,\mu)}} \mu \\ &+ \sum_{o \text{ is a !-link of } ground(\pi)} \sum_{e \in e(o)} \mathcal{W}(e) \ .\end{aligned}$$

*If $c_1, \ldots, c_n$ are the conclusions of $\pi$, then the* result of $e$*, denoted by $|e|$, is the element*[4] *$\langle e(c_1), \ldots, e(c_n)\rangle$ of $⅋^n_{i=1} D$. The* $\llbracket\rrbracket$-interpretation of $\pi$ *is the set of the results of its $\llbracket\rrbracket$-experiments. The* $\llparenthesis\rrparenthesis$-interpretation of $\pi$ *is the set of the pairs $(|e|, \mathcal{W}(e))$ such that $e$ is an $\llparenthesis\rrparenthesis$-experiment of $\pi$.*

$$\begin{aligned}\llbracket\pi\rrbracket \quad &:= \quad \{\langle e(c_1), \ldots, e(c_n)\rangle \,;\, e \text{ is an } \llbracket\rrbracket\text{-experiment of } \pi\} \ ; \\ \llparenthesis\pi\rrparenthesis \quad &:= \quad \{(\langle e(c_1), \ldots, e(c_n)\rangle, \mathcal{W}(e)) \,;\, e \text{ is an } \llparenthesis\rrparenthesis\text{-experiment of } \pi\} \ .\end{aligned}$$

*If $\mathbf{y} = \langle e(c_1), \ldots, e(c_n)\rangle$ is the result of an $\llbracket\rrbracket$-experiment (resp. an $\llparenthesis\rrparenthesis$-experiment) $e$ of $\pi$, we denote by $\mathbf{y}_{c_i}$ the element $e(c_i)$, for every $i \leq n$. Generally, if $\mathbf{d} = \langle c_{i_1}, \ldots, c_{i_k}\rangle$ is a sequence of conclusions of $\pi$, we note by $\mathbf{y}_\mathbf{d}$ the element $\langle e(c_{i_1}), \ldots, e(c_{i_k})\rangle$ of* $\mathbf{D}$.

**Definition 11.** *We define, by induction on depth($\pi$), what means to be* w-sparing *for a $\llparenthesis\rrparenthesis$-experiment $e$ of a net $\pi$:*

[2] Notice, however, that while an $\llbracket\rrbracket$-experiment can associate with a !-link of *ground*($\pi$) an empty multiset of experiments, this cannot be the case for an $\llparenthesis\rrparenthesis$-experiment.

[3] Notice that when $e$ is an $\llbracket\rrbracket$-experiment one always has $\mathcal{W}(e) = []$.

[4] Recall that a g-structure, hence a ♭-net, is given together with an order on its conclusions, so the sequence $\langle e(c_1), \ldots, e(c_n)\rangle$ is uniquely determined by $e$ and $\pi$.

- *for every conclusion c of a weakening-link of ground($\pi$) which is not premise of some cut-link, we have $e(c) = (-, [])$;*

- *for every !-link o of ground($\pi$), $e(o)$ is a finite multiset of w-sparing experiments of $\pi^o$.*

**Proposition 12.** *For $\pi$ and $\pi_1$ nets: if $\pi \rightsquigarrow^*_{(\neg e)_s} \pi_1$, then $(\!|\pi|\!) = (\!|\pi_1|\!)$.*

*Proof.* A straightforward consequence of Lemma 22. □

The following definition introduces an equivalence relation $\sim$ on the $(\!|\,|\!)$-experiments of a $\flat$-net $\pi$: intuitively the $\sim$ equivalence classes are made of experiments associating with a given !-link of $\pi$ multisets of experiments with the same cardinality.

**Definition 13.** *We define an* equivalence $\sim$ *on the set of $(\!|\,|\!)$-experiments of a $\flat$-net $\pi$, by induction on depth($\pi$). Let $e, e' : \pi$, we set $e \sim e'$ whenever*

- *for any weakening-link l of ground($\pi$), we have $e(c)(-, [])$ if, and only if, $e'(c) = (-, [])$, where c is the conclusion of l;*

- *and, for every !-node o of ground($\pi$), there is $m \in \mathbb{N}$ such that $e(o) = [e_1, \ldots, e_m]$, $e'(o) = [e'_1, \ldots, e'_m]$ and, for any $j \in \{1, \ldots, m\}$, we have $e_j \sim e'_j$.*

# 4 Strong Normalization

We now define the function allowing to compute $(\!|\pi|\!)$ from $[\![\pi]\!]$, when $\pi$ is cut-free (Proposition 19). We use the notion of substitution: a function $\sigma : D \to D$ induced by a function $\sigma^A : A \to D$; we denote by $\mathcal{S}$ the set of substitutions.

An important property is that the interpretation of a $\flat$-net is closed by substitution, as the next lemma shows (the proof is an easy induction on $s(\pi)$).

**Lemma 14.** *Let $\pi$ be a net. For every $(\!|\,|\!)$-experiment $e'$ of $\pi$, for every $\sigma \in \mathcal{S}$, there is a $(\!|\,|\!)$-experiment e of $\pi$ such that $(\sigma(|e'|), \sigma(\mathcal{W}(e'))) = (|e|, \mathcal{W}(e))$ and $e \sim e'$.*

**Definition 15.** *We define the function $F : (D^{ex})^n \to \mathcal{P}_f(D^n \times \mathcal{M}_{fin}(D))$ by stating*

$$F(\langle x_1, \ldots, x_n \rangle) = \{(\langle y_1, \ldots, y_n \rangle, \sum_{i=1}^{n} \mathcal{W}_i); (y_1, \mathcal{W}_1) \in F(x_1), \ldots, (y_n, \mathcal{W}_n) \in F(x_n)\}$$

*and $F : D^{ex} \to \mathcal{P}_f(D \times \mathcal{M}_{fin}(D))$*[5] *is defined by induction on the depth of $x$*[6]*:*

- *if $x \in D_0$, then $F(x) = \{(x, [])\}$*

[5]We keep the same notation for $F : (D^{ex})^n \to \mathcal{P}_f(D^n \times \mathcal{M}_{\text{fin}}(D))$ and $F : D^{ex} \to \mathcal{P}_f(D \times \mathcal{M}_{\text{fin}}(D))$.

[6]That is the least number $n \in \mathbb{N}$ s.t. $x \in D_n$.

- *if* $x = (\iota, y, y')$*, then* $F(x) = \{((\iota, z, z'), \mathcal{W}+\mathcal{W}') : (z, \mathcal{W}) \in F(y)$ *and* $(z', \mathcal{W}') \in F(y')\}$
- *if* $x = (+, \beta)$ *where* $\beta = [x_1, \ldots, x_k] \in \mathcal{M}_{fin}(D^{ex})$*, then* $F(x) = \{((+, [x'_1, \ldots, x'_k]), \sum_{i=1}^{k} \mathcal{W}_i) : (x'_i, \mathcal{W}_i) \in F(x_i)\}$[7]
- *if* $x = (-, \beta)$ *where* $\beta = [x_1, \ldots, x_k] \in \mathcal{M}_{fin}(D^{ex})$*, then* $F(x) = \{((-, [x'_1, \ldots, x'_k]), \sum_{i=1}^{k} \mathcal{W}_i) : (x'_i, \mathcal{W}_i) \in F(x_i)\}$ *if* $k > 0$ *and* $F(x) = \{(x, [])\} \cup \{((-, [\alpha]), [\alpha]) : \alpha \in D\}$ *if* $k = 0$.

**Definition 16.** *For any net* $\pi$*, we define, by induction of depth*$(\pi)$*, what means to be* atomic *(resp.* exhaustive*) for any experiment of* $\pi$*:*

- *An experiment of a net* $\pi$ *of depth* $0$ *is said to be* atomic *if it associates with every conclusion of every axiom of ground*$(\pi)$ *an element of* $\{+, -\} \times A$*.*
- *An experiment of a net* $\pi$ *of depth* $n+1$ *is said to be* atomic *if*
  - *it associates with every conclusion of every axiom of ground*$(\pi)$ *an element of* $\{+, -\} \times A$
  - *and it associates with every* $!$*-link* $o$ *of ground*$(\pi)$ *a finite multiset of atomic experiments of* $\pi^o$*.*
- *A* $[\![]\!]$*-experiment* $e$ *of a net* $\pi$ *is* exhaustive *when* $|e| \in (D^{ex})^n$ *for some* $n \geq 0$*.*

The following definition allows in particular to define the subset $[\![\pi]\!]_{At}$ of the "atomic" elements of $[\![\pi]\!]$, which will be used in Proposition 19.

**Definition 17.** *Given* $E \in \mathfrak{P}(D^n)$ *for some* $n \geq 1$*, we say that* $r \in E$ *is* $E$-atomic *when for every* $r' \in E$ *and every substitution* $\sigma$ *such that* $\sigma(r') = r$ *one has* $\sigma(\gamma) \in \{+, -\} \times A$ *for every* $\gamma \in \{+, -\} \times A$ *that occurs in* $r'$*. For* $E \in \mathfrak{P}(D^n)$*, we denote by* $E_{At}$ *the subset of* $E$ *consisting of the* $E$*-atomic elements.*

**Lemma 18.** *Let* $\pi$ *be a cut free net. Then* $\{(|e|, \mathcal{W}(e)); e$ *is an atomic* $(\!|\,|\!)$*-experiment of* $\pi\} = \bigcup_{\boldsymbol{x} \in ([\![\pi]\!]_{At})^{ex}} F(\boldsymbol{x})$*.*

*Proof.* To prove the inclusion $\bigcup_{\mathbf{x} \in ([\![\pi]\!]_{At})^{ex}} F(\mathbf{x}) \subseteq \{(|e|, \mathcal{W}(e)); e$ is an atomic $(\!|\,|\!)$-experiment of $\pi\}$, we prove, by induction on $s(\pi)$, that, for every exhaustive atomic $[\![]\!]$-experiment $e$ of $\pi$ and for every $(y, \mathcal{W}) \in F(|e|)$, there exists an atomic $(\!|\,|\!)$-experiment $e'$ of $\pi$ such that $(|e'|, \mathcal{W}(e')) = (y, \mathcal{W})$.

To prove the inclusion $\{(|e|, \mathcal{W}(e))$ ; $e$ is an atomic $(\!|\,|\!)$-experiment of $\pi\} \subseteq \bigcup_{\mathbf{x} \in ([\![\pi]\!]_{At})^{ex}} F(\mathbf{x})$, we prove, by induction on $s(\pi)$, that, for every atomic $(\!|\,|\!)$-experiment $e'$ of $\pi$, there exists an exhaustive atomic $[\![]\!]$-experiment $e$ of $\pi$ such that $(|e'|, \mathcal{W}(e')) \in F(|e|)$. $\square$

**Proposition 19.** *Let* $\pi$ *be a cut free net. Then* $(\!|\pi|\!) = \{(\sigma(\boldsymbol{y}), \sigma(\mathcal{W}))$ ; $(\boldsymbol{y}, \mathcal{W}) \in \bigcup_{\boldsymbol{x} \in ([\![\pi]\!]_{At})^{ex}} F(\boldsymbol{x})$ *and* $\sigma \in \mathcal{S}\}$*.*

[7]Notice that since $x \in D^{\mathrm{ex}}$, one has $k \geq 1$.

We extend the definition of size of an $[\![\,]\!]$-experiment given in [6] to $(\!|\,|\!)$-experiments, and we introduce a new notion of size of $(\!|\,|\!)$-experiments. This new notion of size is crucial to establish our main results (see Lemma 22).

**Definition 20** (Size of experiments)**.** *For every $\flat$-net $\pi$, for every $(\!|\,|\!)$-experiment $e$ of $\pi$, we define, by induction on $depth(\pi)$, the* size of $e$, *$s(e)$ for short, as follows:*

$$s(e) = s(ground(\pi)) + \sum_{o \in !(ground(\pi))} \sum_{e^o \in e(o)} s(e^o) \ .$$

*We set $s_{(\!|\,|\!)}(e) = s(e) + 2\,Card(\mathcal{W}(e))$.*

**Remark 1.** *A 1-$(\!|\,|\!)$-experiment $e$ of a $\flat$-net $\pi$ is a $(\!|\,|\!)$-experiment such that, for any !-link $o$ of $ground(\pi)$, we have $e(o) = [e_1]$ with $e_1$ a 1-$(\!|\,|\!)$-experiment of $\pi^o$. When such an $(\!|\,|\!)$-experiment $e$ of $\pi$ exists, we have $s(\pi) = s(e) = \min\{s(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi\}$.*

**Fact 21.** *If $e$ and $e'$ are two -$(\!|\,|\!)$-experiments of a $\flat$-net $\pi$, then from $e \sim e'$ it follows that $s_{(\!|\,|\!)}(e) = s(e')$.*

The following lemma plays, in the framework of strong normalization, a similar role as Lemma 20 of [6].

**Lemma 22.** *Let $\pi$ and $\pi_1$ be two nets such that $\pi \rightsquigarrow_{(\neg e)_s} \pi_1$. Then*

1. *for every $(\!|\,|\!)$-experiment $e$ of $\pi$ such that $s_{(\!|\,|\!)}(e) = \min\{s_{(\!|\,|\!)}(e); e$ is an $(\!|\,|\!)$-experiment of $\pi\}$, there exists an $(\!|\,|\!)$-experiment $e_1$ of $\pi_1$ such that $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$ and $s_{(\!|\,|\!)}(e_1) = s_{(\!|\,|\!)}(e) - 2$;*

2. *for every $(\!|\,|\!)$-experiment $e_1$ of $\pi_1$ such that $s_{(\!|\,|\!)}(e_1) = \min\{s_{(\!|\,|\!)}(e); e$ is an $(\!|\,|\!)$-experiment of $\pi_1\}$, there exists an $(\!|\,|\!)$-experiment $e$ of $\pi$ such that $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$ and $s_{(\!|\,|\!)}(e_1) = s_{(\!|\,|\!)}(e) - 2$.*

*Proof.* The fact that $|e| = |e_1|$ is completely standard. Furthermore the reduction step leading from $\pi$ to $\pi_1$ is non erasing, which implies $\mathcal{W}(e) = \mathcal{W}(e_1)$. By Definition 20, it then remains to prove that $s(e) = s(e_1) - 2$, which for stratified steps has been proven in [6]. □

**Theorem 23.** *A net $\pi$ is strongly normalizable if, and only, if $(\!|\pi|\!)$ is non-empty.*

*Proof.* By Proposition 6, it is enough to show that, for any net $\pi$, we have $\pi \in \mathbf{SN}^{(\neg e)_s}$ if, and only if, $(\!|\pi|\!)$ is non-empty. If $\pi \in \mathbf{SN}^{(\neg e)_s}$, let $\pi \rightsquigarrow^*_{(\neg e)_s} \pi_0$ with $\pi_0$ $\neg e$-normal. There obviously exists a 1-$(\!|\,|\!)$-experiment $e$ of $\pi_0$, and thus $|e| \in (\!|\pi|\!)$ (by Proposition 12).
Conversely, one proves by induction on $\min\{s_{(\!|\,|\!)}(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi\}$ that $\pi \in \mathbf{SN}^{(\neg e)_s}$. If $\pi$ is $\neg e$-normal, we are done. Otherwise, we show that for every $\pi_1$ such that $\pi \rightsquigarrow_{(\neg e)_s} \pi_1$, one has $\pi_1 \in \mathbf{SN}^{(\neg e)_s}$. Since $(\!|\pi|\!) \neq \emptyset$, there exist $(\!|\,|\!)$-experiments of $\pi$ and we can select $e$ such that $s_{(\!|\,|\!)}(e) = \min\{s_{(\!|\,|\!)}(e); e$ is an $(\!|\,|\!)$-experiment of $\pi\}$. By Lemma 22, there exists a $(\!|\,|\!)$-experiment $e_1$ of $\pi_1$ such that $s_{(\!|\,|\!)}(e_1) = s_{(\!|\,|\!)}(e) - 2$, and still by Lemma 22, $s_{(\!|\,|\!)}(e_1) = \min\{s_{(\!|\,|\!)}(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi_1\}$: by induction hypothesis $\pi_1 \in \mathbf{SN}^{(\neg e)_s}$. □

**Corollary 24.** *Let $\pi$ (resp. $\pi'$) be a net with conclusions $\boldsymbol{d}, c$ (resp. $\mathbf{d}', c'$). The net $(\pi|\pi')_{c,c'}$ is strongly normalizable if, and only if, there are $(\mathbf{x}, \mathcal{W}) \in (\!|\pi|\!)$ and $(\mathbf{x}', \mathcal{W}') \in (\!|\pi'|\!)$ such that $\mathbf{x}_c = {\mathbf{x}'_{c'}}^{\perp}$.*

Now, it is easy to see that for a $\neg e$-normal net $\pi$, the longest reduction sequence starting from $\pi$ has a number of (erasing) steps equal to the number of (erasing) cuts of $\pi$. And the following lemma shows how this number can be computed from a (well chosen) element of $(\!|\pi|\!)$: the second component of the pair $(|e|, \mathcal{W}) \in (\!|\pi|\!)$, where $e$ is an $(\!|\,|\!)$-experiment of $\pi$ with minimal size.

**Lemma 25.** *Let $\pi$ be a $\neg e$-normal net. Let $e_0$ be an $(\!|\,|\!)$-experiment of $\pi$ such that*

$$s_{(\!|\,|\!)}(e_0) = \min\{s_{(\!|\,|\!)}(e) \; ; \; e \textit{ is an } (\!|\,|\!)\textit{-experiment of } \pi\}.$$

*Then $Card(\mathcal{W}(e_0))$ is the number of cuts of $\pi$.*

*Proof.* Given a net $\pi$, if there exists a $w$-sparing 1-$(\!|\,|\!)$-experiment $e_1$ of $\pi$, then

- any $(\!|\,|\!)$-experiment $e_0$ such that $s_{(\!|\,|\!)}(e_0) = \min\{s_{(\!|\,|\!)}(e); e$ is an $(\!|\,|\!)$-experiment of $\pi\}$ is a $w$-sparing 1-$(\!|\,|\!)$-experiment
- and $Card(\mathcal{W}(e_1))$ is the number of cuts of $\pi$.

Now, since $\pi$ is $\neg e$-normal, there exists such an $(\!|\,|\!)$-experiment $e_1$. □

To conclude, (using the notations of Corollary 8), we have to compute the length of the two sequences $R_1$ and $R_2$ of Corollary 8 from $[\![\pi]\!]$ and $[\![\pi']\!]$. Since one can compute $(\!|\pi|\!)$ and $(\!|\pi'|\!)$ from $[\![\pi]\!]$ and $[\![\pi']\!]$ (thanks to Proposition 19), it is enough to compute the length of $R_1$ and $R_2$ from $(\!|\pi|\!)$ and $(\!|\pi'|\!)$. We need for this a notion of size of an element of the $(\!|\,|\!)$-interpretation of a net, which is a particular case of size of an element of $D^n \times \mathcal{M}_{\text{fin}}(D)$. Like for the notion of size of an experiment, we use the notion of size of an element of $D$ introduced in [6]:

**Definition 26** (Size of elements)**.** *For every $x \in D$, we define the* size $s(x)$ *of $x$, by induction on depth$(x)$. Let $p \in \{+,-\}$,*
- *if $x \in \{+,-\} \times A$ or $x = (p, *)$, then $s(x) = 1$;*
- *if $x = (p, y, z)$, then $s(x) = 1 + s(y) + s(z)$;*
- *if $x = (p, [x_1, \ldots, x_m])$, then $s(x) = 1 + \sum_{j=1}^{m} s(x_j)$;*

*Given $(x_1, \ldots, x_n) \in \bigparr_{i=1}^{n} D$ ($n \geq 0$), we set $s(x_1, \ldots, x_n) = \sum_{i=1}^{n} s(x_i)$ and $s([x_1, \ldots, x_n]) = \sum_{i=1}^{n} s(x_i)$*[8].

*Let $n \geq 1$ and $(\boldsymbol{x}, \mathcal{W}) \in D^n \times \mathcal{M}_{fin}(D)$. Then we set $s_{(\!|\,|\!)}(\boldsymbol{x}, \mathcal{W}) = s(\boldsymbol{x}) + \sum_{\alpha \in D} \mathcal{W}(\alpha) \cdot (s(\alpha) + 2)$.*

We first compute the length of $R_1$ by means of experiments (Proposition 29).

**Definition 27.** *Let $n \geq 1$. For any $X \subseteq D^n \times \mathcal{M}_{fin}(D)$, we set $s_{(\!|\,|\!)\,inf}(X) = \inf\{s_{(\!|\,|\!)}(x); x \in X\}$.*

[8] Notice that for every point $x \in D$ or $x \in \bigparr_{i=1}^{n} D \cup D^{<\infty} \cup \mathcal{M}_{\text{fin}}(D)$, $s(x)$ is the number of occurrences of $+$, $-$ in $x$ (seen as a word).

**Lemma 28.** *Let $\pi$ be a $\flat$-net with $k$ structural conclusions. If $\pi$ is $\neg e$-normal, then we have $s_{(\!|\,|\!)_{inf}}((\!|\pi|\!)) = s(\pi) + k = \min\{s(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi\} + k$.*

*Proof.* We consider the cut free net $\pi'$ obtained from $\pi$ in two steps:

- first, we erase all the weakening-links premises of some cut-link and all the cut-links;
- second, under every !-link whose conclusion was premise of some cut-link, we add a $\flat$-link and a unary contraction at depth 0 under this $\flat$-link.

First, notice that we have $s(\pi') = s(\pi)$. Second, notice that, for any $w$-sparing 1-$(\!|\,|\!)$-experiment $e$ of $\pi$, the 1-experiment $e'$ of $\pi'$ induced by $e$ enjoys the following property: $s(|e'|) = s_{(\!|\,|\!)}(|e|, \mathcal{W}(e))$.

Now, since $\pi$ is $\neg e$-normal, we can define, by induction on $depth(\pi)$, a $w$-sparing $1 - (\!|\,|\!)$-experiment $e_1 : \pi$ that associates $(p, *)$ with the conclusions of axiom nodes. More precisely, $e_1$ is defined as follows:

- with every conclusion of a weakening of $ground(\pi)$ that is premise of some cut, $e_1$ associates the element $(-, [\alpha^\perp])$, where $\alpha$ is such that $e_1$ associates $(+, [\alpha])$ with the other premise of the cut;
- with every pair of conclusions of every $ax$-link of $ground(\pi)$, $e_1$ associates the pair of elements $(+, *)$, $(-, *)$ (it does not matter in which order);
- with every !-link $o$, $e_1$ associates the singleton $[e_1^o]$, where $e_1^o$ is an experiment defined as $e_1$ on $\pi^o$ (notice that $depth(\pi^o) < depth(\pi)$).

We denote by $e_1'$ the 1-experiment of $\pi'$ induced by $e_1$: we have $s(|e_1'|) = s(\pi') + k$ (induction on $depth(\pi')$) and $s(|e_1'|) = s_{(\!|\,|\!)}(|e_1|, \mathcal{W}(e_1))$, hence $s(\pi) + k = s(\pi') + k = s_{(\!|\,|\!)}(|e_1|, \mathcal{W}(e_1))$. By Remark 1, we have $s(\pi) = s(e_1) = \min\{s(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi\}$. Lastly, since $e_1$ is a $w$-sparing atomic 1-experiment of $\pi$ that associates $(p, *)$ with the conclusions of axiom nodes, we have $s_{(\!|\,|\!)}(|e_1|, \mathcal{W}(e_1)) = s_{(\!|\,|\!)_{inf}}((\!|\pi|\!))$. $\square$

**Proposition 29.** *Let $\pi$ be a net and let $\pi'$ be a $\neg e$-normal net. For every reduction sequence $R : \pi \rightsquigarrow^*_{(\neg e)_s} \pi'$, and every $(\!|\,|\!)$-experiment $e_0$ of $\pi$ such that $s_{(\!|\,|\!)}(e_0) = \min\{s_{(\!|\,|\!)}(e)\,;\, e$ is an $(\!|\,|\!)$-experiment of $\pi\}$, we have $length(R) = (s(e_0) - s_{(\!|\,|\!)_{inf}}((\!|\pi|\!)))/2$.*

*Proof.* By induction on $length(R)$. If $length(R) = 0$, apply Lemma 28.

Now, $R = \pi \rightsquigarrow^*_{(\neg e)_s} \pi_1 \rightsquigarrow^*_{(\neg e)_s} \pi'$. By Lemma 22, there is an $(\!|\,|\!)$-experiment $e_1$ of $\pi_1$ such that $(|e_1|, \mathcal{W}(e_1)) = (|e_0|, \mathcal{W}(e_0))$, $s_{(\!|\,|\!)}(e_1) = s_{(\!|\,|\!)}(e_0) - 2$ and $s_{(\!|\,|\!)}(e_1) = \min\{s_{(\!|\,|\!)}(e)\,;\, e :_{(\!|\,|\!)} \pi_1\}$.

We have $s(e_0) - s(e_1) = s_{(\!|\,|\!)}(e_0) - s_{(\!|\,|\!)}(e_1) = 2$.

We apply the induction hypothesis to $\pi_1$. We have $length(R) - 1 = (s(e_1) - s_{(\!|\,|\!)_{inf}}((\!|\pi_1|\!)))/2 = (s(e_1) - s_{(\!|\,|\!)_{inf}}((\!|\pi|\!)))/2 = (s(e_0) - 2 - s_{(\!|\,|\!)_{inf}}((\!|\pi|\!)))/2$ $\square$

The following lemma shows that if $\pi$ is cut free and has no structural conclusions and $e$ is an $\llbracket\rrbracket$-experiment of $\pi$, then $s(e) \leq s(|e|) - s(\mathcal{W}(e))$:

**Lemma 30.** *Let $\pi$ be a cut free $\flat$-net with $k$ structural conclusions and let $e$ be a $\llbracket\rrbracket$-experiment of $\pi$. Then we have $s(e) \leq s(|e|) - s(\mathcal{W}(e)) - k$.*

*Proof.* The proof is by induction on $s(\pi)$. If *ground*$(\pi)$ is an axiom, then $k = 0$ and $s(\mathcal{W}(e)) = 0$: if the elements of $D$ associated with the conclusions of the axiom are of the shape $(p, a)$ with $a \in A \cup \{*\}$, then we have $s(e) = s(|e|)$; else, we have $s(e) < s(|e|)$. Now, assume that *ground*$(\pi)$ is a !-link $o$ with $k$ structural conclusions. Set $e(o) = [e_1, \ldots, e_m]$ with $m \geq 1$ and let $\pi^o$ be the box of $o$. Notice that $\pi$ has $k+1$ conclusions. We have

$$\begin{array}{rcll} s(e) & = & 1 + \sum_{j=1}^{m} s(e_j) & \\ & \leq & 1 + \sum_{j=1}^{m} (s(|e_j|) - s(\mathcal{W}(e_j)) - k) & \text{(by induction hypothesis)} \\ & = & 1 + s(|e|) - s(\mathcal{W}(e)) - (k+1) & \\ & = & s(|e|) - s(\mathcal{W}(e)) - k. & \end{array}$$

The other cases are left to the reader. □

**Lemma 31.** *Assume $A$ is infinite. Let $\pi$ be a cut free $\flat$-net with $k$ structural conclusions (and possibly other logical conclusions), and let $e$ be an $\llbracket\rrbracket$-experiment of $\pi$. There exist $e' \sim e$ and a substitution $\sigma$ such that $s(e') = s(|e'|) - s(\mathcal{W}(e')) - k$ and $\sigma(|e'|, \mathcal{W}(e')) = (|e|, \mathcal{W}(e))$.*

*Proof.* Let $A_0$ be the set of elements of $A$ occurring in $\mathcal{W}(e)$. We prove, by induction on $s(\pi)$, that, for every infinite subset $A'$ of $A \setminus A_0$, there is an experiment $e' \sim e$ such that

1. $s(e') = s(|e'|) - s(\mathcal{W}(e')) - k$;
2. $\sigma(|e'|, \mathcal{W}(e')) = (|e|, \mathcal{W}(e))$ for some $\sigma \in \mathcal{S}$ such that $\sigma_{\restriction A_0} = id_{A_0}$;
3. and every element of $A \setminus A_0$ occurring in $|e'|$ is an element of $A'$.

In the case *ground*$(\pi)$ is a weakening-link $l$, we set $e'(c) = e(c)$, where $c$ is $l$'s conclusion. The other cases are similar to the proof of Lemma 35 of [6]. □

**Lemma 32.** *Assume $A$ is infinite. Let $\pi$ be a cut-free net and let $e$ be an $\llbracket\rrbracket$-experiment of $\pi$. We have $s_{\llbracket\rrbracket}(e) = \min\{s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e'));\ e' \sim e \text{ and } (\exists \sigma \in \mathcal{S})\sigma(|e'|, \mathcal{W}(e')) = (|e|, \mathcal{W}(e))\}$.*

*Proof.* We set $q = \min\{s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e'));\ e' \sim e$ and $(\exists \sigma \in \mathcal{S})\sigma(|e'|, \mathcal{W}(e')) = (|e|, \mathcal{W}(e))\}$.

First, we prove $s_{\llbracket\rrbracket}(e) \leq q$. Let $e'_0$ be an $\llbracket\rrbracket$-experiment of $\pi$ such that $e'_0 \sim e$ and $s(|e'_0|) - s(\mathcal{W}(e'_0)) + 2\,Card(\mathcal{W}(e'_0)) = q$. By Fact 21 and Lemma 30, we have

$s_{\llbracket\rrbracket}(e) = s_{\llbracket\rrbracket}(e'_0) = s(e'_0) + 2\,Card(\mathcal{W}(e'_0)) \leq s(|e'_0|) - s(\mathcal{W}(e'_0)) + 2\,Card(\mathcal{W}(e'_0)) = q$.

Now, we prove $q \leq s_{\llbracket\rrbracket}(e)$. By Lemma 31, there exist $e' \sim e$ and a substitution $\sigma$ such that $s(e') = s(|e'|) - s(\mathcal{W}(e'))$, $\sigma(|e'|) = |e|$ and $\sigma(\mathcal{W}(e')) = \mathcal{W}(e)$. We have $q \leq s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e')) = s(e') + 2\,Card(\mathcal{W}(e')) = s_{\llbracket\rrbracket}(e') = s_{\llbracket\rrbracket}(e)$ (again by Fact 21). $\square$

**Proposition 33.** *Assume $A$ infinite. Let $\pi$ be a cut-free net and let $(\mathbf{x}, \mathcal{V}) \in \llbracket\pi\rrbracket$.*
*We have* $\min\{s_{\llbracket\rrbracket}(e); e$ *is an* $\llbracket\rrbracket$*-experiment of* $\pi$ *such that* $(|e|, \mathcal{W}(e)) = (\mathbf{x}, \mathcal{V})\}$

$$= \min\left\{ s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e'))\ ;\ \begin{array}{l} e' \textit{ is an } \llbracket\rrbracket\textit{-experiment of } \pi \textit{ such that} \\ (\exists \sigma \in \mathcal{S})\ \sigma(|e'|, \mathcal{W}(e')) = (\mathbf{x}, \mathcal{V}) \end{array} \right\}.$$

*Proof.* Set $r = \min\left\{ s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e'))\ ;\ \begin{array}{l} e' \text{ is an } \llbracket\rrbracket\text{-experiment of } \pi \text{ such that} \\ (\exists \sigma \in \mathcal{S})\ \sigma(|e'|, \mathcal{W}(e')) = (\mathbf{x}, \mathcal{V}) \end{array} \right\}$
and
$q = \min\{s_{\llbracket\rrbracket}(e)\ ;\ e$ is an $\llbracket\rrbracket$-experiment of $\pi$ such that $(|e|, \mathcal{W}(e)) = (\mathbf{x}, \mathcal{V})\}$.

First we prove $q \leq r$. Let $e'_0$ be an $\llbracket\rrbracket$-experiment of $\pi$ such that

- $(\exists \sigma \in \mathcal{S})\ \sigma(|e'_0|, \mathcal{W}(e'_0)) = (\mathbf{x}, \mathcal{V})$
- and $s(|e'_0|) - s(\mathcal{W}(e'_0)) + 2\,Card(\mathcal{W}(e'_0)) = r$.

By Fact 21 and Lemma 14, there exists an $\llbracket\rrbracket$-experiment $e_0$ of $\pi$ such that $|e_0| = (\mathbf{x}, \mathcal{V})$ and $s_{\llbracket\rrbracket}(e_0) = s_{\llbracket\rrbracket}(e'_0)$. By Lemma 30, we have $q \leq s_{\llbracket\rrbracket}(e_0) = s_{\llbracket\rrbracket}(e'_0) = s(e'_0) + 2\,Card(\mathcal{W}(e'_0)) \leq s(|e'_0|) - s(\mathcal{W}(e'_0)) + 2\,Card(\mathcal{W}(e'_0)) = r$.

Now, we prove $r \leq q$. Let $e$ be an $\llbracket\rrbracket$-experiment of $\pi$ such that $s_{\llbracket\rrbracket}(e) = q$ and $(|e|, \mathcal{W}(e)) = (\mathbf{x}, \mathcal{V})$. By Lemma 32, we have $s_{\llbracket\rrbracket}(e) = \min\{s(|e'|) - s(\mathcal{W}(e')) + 2\,Card(\mathcal{W}(e')); e' \sim e$ and $(\exists \sigma \in \mathcal{S})\sigma(|e'|, \mathcal{W}(e')) = (|e|, \mathcal{W}(e))\} \geq r$. $\square$

We now state our main theorem: still using the notations of Corollary 8, we show how to compute the length of $R_1$ and $R_2$ from $\llbracket\pi\rrbracket$ and $\llbracket\pi'\rrbracket$.

**Theorem 34.** *Assume $A$ infinite. Let $\pi$ and $\pi'$ be two cut-free nets with conclusions $\mathbf{d}, c$ (resp. $\mathbf{d'}, c'$). The value of strong$((\pi|\pi')_{c,c'})$ is*

$$\inf \left\{ \begin{array}{l} \frac{s_{\llbracket\rrbracket}(\mathbf{z},\mathcal{W}) + s_{\llbracket\rrbracket}(\mathbf{z}',\mathcal{W}') - s_{\llbracket\rrbracket_{inf}}(\llbracket(\pi|\pi')_{c,c'}\rrbracket)}{2} - s(\mathcal{W} + \mathcal{W}'); \\ (\mathbf{z}, \mathcal{W}) \in \llbracket\pi\rrbracket, (\mathbf{z}', \mathcal{W}') \in \llbracket\pi'\rrbracket \textit{ and } (\exists \sigma \in \mathcal{S})\ \sigma(\mathbf{z}_c) = \sigma(\mathbf{z}'_{c'})^{\perp} \end{array} \right\}$$

*Proof.* We set

$$\mathcal{C} = \left\{ \begin{array}{l} \frac{s_{\llbracket\rrbracket}(\mathbf{z},\mathcal{W}) + s_{\llbracket\rrbracket}(\mathbf{z}',\mathcal{W}') - s_{\llbracket\rrbracket_{inf}}(\llbracket(\pi|\pi')_{c,c'}\rrbracket)}{2} - s(\mathcal{W} + \mathcal{W}'); \\ (\mathbf{z}, \mathcal{W}) \in \llbracket\pi\rrbracket, (\mathbf{z}', \mathcal{W}') \in \llbracket\pi'\rrbracket \text{ and } (\exists \sigma \in \mathcal{S})\ \sigma(\mathbf{z}_c) = \sigma(\mathbf{z}'_{c'})^{\perp} \end{array} \right\}.$$

In the case where $(\pi|\pi')_{c,c'}$ is not strongly normalizable, by Corollary 24 and Lemma 14, we have $\mathcal{C} = \emptyset$.

Now, we assume that $(\pi|\pi')_{c,c'}$ is strongly normalizable.

By Corollary 8, there exist $R_1 : (\pi|\pi')_{c,c'} \rightsquigarrow^*_{(\neg e)_s} \pi_1$ and $R_2 : \pi_1 \rightsquigarrow_e{}^* \pi_2$ such that

- $\pi_1$ is $\neg e$-normal;
- $\pi_2$ is cut-free;
- and $\mathit{strong}((\pi|\pi')_{c,c'}) = \mathit{length}(R_1) + \mathit{length}(R_2)$.

By Corollary 24, there are $(\mathbf{x},\mathcal{V}) \in [\![\pi]\!]$ and $(\mathbf{x}',\mathcal{V}') \in [\![\pi']\!]$ such that $\mathbf{x}_c = {\mathbf{x}'_{c'}}^{\perp}$. Hence the set $\mathcal{C}$ is non-empty. We can thus consider some $(\mathbf{z},\mathcal{W}) \in [\![\pi]\!], (\mathbf{z}',\mathcal{W}') \in [\![\pi']\!]$ and $\sigma \in \mathcal{S}$ such that $\sigma(\mathbf{z}_c) = \sigma(\mathbf{z}'_{c'})^{\perp}$ and $\frac{s_{(\!)}(z,\mathcal{W})+s_{(\!)}(z',\mathcal{W}')-s_{(\!)\,inf}([\![(\pi|\pi')_{c,c'}]\!])}{2} - s(\mathcal{W}+\mathcal{W}') = \min(\mathcal{C})$. We set $\mathbf{x} = \sigma(\mathbf{z})$, $\mathbf{x}' = \sigma(\mathbf{z}')$, $\mathcal{V} = \sigma(\mathcal{W})$ and $\mathcal{V}' = \sigma(\mathcal{W}')$.
By Lemma 14, we have $(\mathbf{x},\mathcal{V}) \in [\![\pi]\!]$ and $(\mathbf{x}',\mathcal{V}') \in [\![\pi']\!]$. Since $\mathbf{x}_c = {\mathbf{x}'_{c'}}^{\perp}$, there exists a $(\!)$-experiment $e_0$ of $(\pi|\pi')_{c,c'}$ such that

- $\mathcal{W}(e_0) = \mathcal{V} + \mathcal{V}'$;
- and

$$\begin{array}{rcl} s_{(\!)}(e_0) & = & \min\{s_{(\!)}(e)\,;\, e \text{ is an } (\!)\text{-experiment of } \pi \text{ such that } (|e|,\mathcal{W}(e)) = (\mathbf{x},\mathcal{V})\} + \\ & & \min\{s_{(\!)}(e')\,;\, e' \text{ is an } (\!)\text{-experiment of } \pi' \text{ such that } (|e'|,\mathcal{W}(e')) = (\mathbf{x}',\mathcal{V}')\}. \end{array}$$

By applying Proposition 33 twice, we obtain

$$\begin{array}{rcl} s_{(\!)}(e_0) & = & \min\left\{ s(\mathbf{z}) - s(\mathcal{W}) + 2\,\mathit{Card}(\mathcal{W})\,;\, \begin{array}{l} (\mathbf{z},\mathcal{W}) \in [\![\pi]\!] \text{ such that} \\ (\exists\sigma \in \mathcal{S})\ (\sigma(\mathbf{z}),\sigma(\mathcal{W})) = (\mathbf{x},\mathcal{V}) \end{array} \right\} + \\ & & \min\left\{ s(\mathbf{z}') - s(\mathcal{W}') + 2\,\mathit{Card}(\mathcal{W}')\,;\, \begin{array}{l} (\mathbf{z}',\mathcal{W}') \in [\![\pi']\!] \text{ such that} \\ (\exists\sigma \in \mathcal{S})\ (\sigma(\mathbf{z}'),\sigma(\mathcal{W}')) = (\mathbf{x}',\mathcal{V}') \end{array} \right\} \\ & = & \min\left\{ \begin{array}{l} s(\mathbf{z}) - s(\mathcal{W}) \\ +s(\mathbf{z}') - s(\mathcal{W}') \\ +2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') \end{array} ;\ \begin{array}{l} (\mathbf{z},\mathcal{W}) \in [\![\pi]\!], (\mathbf{z}',\mathcal{W}') \in [\![\pi']\!] \text{ and} \\ \text{there exists } \sigma \in \mathcal{S} \text{ such} \\ \text{that } \sigma(\mathbf{z},\mathcal{W}) = (\mathbf{x},\mathcal{V}) \\ \text{and } \sigma(\mathbf{z}',\mathcal{W}') = (\mathbf{x}',\mathcal{V}') \end{array} \right\} \\ & & \text{(the points of } [\![\pi]\!] \text{ and } [\![\pi']\!] \text{ we look for are among those with disjoint atoms).} \end{array}$$

Therefore we have $s_{(\!)}(e_0) \leq s(\mathbf{z}) - s(\mathcal{W}) + s(\mathbf{z}') - s(\mathcal{W}') + 2\,\mathit{Card}(\mathcal{W}+\mathcal{W}')$.
Now, we have

$$\begin{array}{rl} & s(\mathbf{z}) - s(\mathcal{W}) + s(\mathbf{z}') - s(\mathcal{W}') + 2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') \\ = & 2(\dfrac{s_{(\!)}(\mathbf{z},\mathcal{W}) + s_{(\!)}(\mathbf{z}',\mathcal{W}') - s_{(\!)\,inf}([\![(\pi|\pi')_{c,c'}]\!])}{2} - s(\mathcal{W}+\mathcal{W}')) + s_{(\!)\,inf}([\![(\pi|\pi')_{c,c'}]\!]) \quad ; \end{array}$$

remember that $\frac{s_{(\!)}(z,\mathcal{W})+s_{(\!)}(z',\mathcal{W}')-s_{(\!)\,inf}([\![(\pi|\pi')_{c,c'}]\!])}{2} - s(\mathcal{W}+\mathcal{W}') = \min(\mathcal{C})$, hence

$$\begin{array}{rl} & s(\mathbf{z}) - s(\mathcal{W}) + s(\mathbf{z}') - s(\mathcal{W}') + 2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') \\ = & \min\left\{ s(\mathbf{z}) - s(\mathcal{W}) + s(\mathbf{z}') - s(\mathcal{W}') + 2\,\mathit{Card}(\mathcal{W}+\mathcal{W}')\,;\ \begin{array}{c} (\mathbf{z},\mathcal{W}) \in [\![\pi]\!], \\ (\mathbf{z}',\mathcal{W}') \in [\![\pi']\!] \\ \text{and} \\ (\exists\sigma \in \mathcal{S})\ \sigma(\mathbf{z}_c) = \sigma(\mathbf{z}'_{c'})^{\perp} \end{array} \right\} \end{array}$$

$$
\begin{aligned}
&= \min\left\{ \begin{array}{l} s(\mathbf{z})-s(\mathcal{W}) \\ +s(\mathbf{z}')-s(\mathcal{W}') \\ +2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') \end{array} ; \begin{array}{l} (\mathbf{z},\mathcal{W})\in (\!|\pi|\!), (\mathbf{z}',\mathcal{W}')\in (\!|\pi'|\!) \text{ and} \\ \text{there exist } (\mathbf{x},\mathcal{V}),(\mathbf{x}',\mathcal{V}'),\sigma\in\mathcal{S} \text{ such} \\ \text{that } \sigma(\mathbf{z},\mathcal{W})=(\mathbf{x},\mathcal{V}), \sigma(\mathbf{z}',\mathcal{W}')=(\mathbf{x}',\mathcal{V}') \\ \text{and } \mathbf{x}_c = {\mathbf{x}'}^{\perp}_{c'} \end{array} \right\} \\
&= \min\left\{ \begin{array}{l} s(\mathbf{z})-s(\mathcal{W}) \\ +s(\mathbf{z}')-s(\mathcal{W}') \\ +2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') \end{array} ; \begin{array}{l} (\mathbf{z},\mathcal{W})\in (\!|\pi|\!), (\mathbf{z}',\mathcal{W}')\in (\!|\pi'|\!) \text{ and} \\ \text{there exist } (\mathbf{x},\mathcal{V})\in (\!|\pi|\!),(\mathbf{x}',\mathcal{V}')\in (\!|\pi'|\!),\sigma\in\mathcal{S} \text{ such} \\ \text{that } \sigma(\mathbf{z},\mathcal{W})=(\mathbf{x},\mathcal{V}), \sigma(\mathbf{z}',\mathcal{W}')=(\mathbf{x}',\mathcal{V}') \\ \text{and } \mathbf{x}_c = {\mathbf{x}'}^{\perp}_{c'} \end{array} \right\} \\
&\quad \text{(by Lemma 14)} \\
&= \min\left\{ \begin{array}{l} \min\{s_{(\!|\,|\!)}(e)\,;\, e \text{ is an } (\!|\,|\!)\text{-experiment of } \pi \text{ such that } (|e|,\mathcal{W}(e))=(\mathbf{x},\mathcal{V})\}+ \\ \min\{s_{(\!|\,|\!)}(e')\,;\, e' \text{ is an } (\!|\,|\!)\text{-experiment of } \pi' \text{ such that } (|e'|,\mathcal{W}(e'))=(\mathbf{x}',\mathcal{V}')\}; \\ (\mathbf{x},\mathcal{V})\in (\!|\pi|\!),(\mathbf{x}',\mathcal{V}')\in (\!|\pi'|\!) \text{ and } \mathbf{x}_c = {\mathbf{x}'}^{\perp}_{c'} \end{array} \right\} \\
&\quad \text{(by applying Proposition 33 twice)} \\
&= \min\left\{ s_{(\!|\,|\!)}(e)+s_{(\!|\,|\!)}(e')\,; \begin{array}{l} e \text{ is an } (\!|\,|\!)\text{-experiment of } \pi, e' \text{ is an } (\!|\,|\!)\text{-experiment of } \pi' \\ \text{and } (\exists (\mathbf{x},\mathcal{V})\in (\!|\pi|\!),(\mathbf{x}',\mathcal{V}')\in (\!|\pi'|\!)) \\ ((|e|,\mathcal{W}(e))=(\mathbf{x},\mathcal{V}) \text{ and } (|e'|,\mathcal{W}(e'))=(\mathbf{x}',\mathcal{V}') \text{ and } \mathbf{x}_c = {\mathbf{x}'_{c'}}^{\perp}) \end{array} \right\} \\
&= \min\{s_{(\!|\,|\!)}(e)\,;\, e \text{ is an } (\!|\,|\!)\text{-experiment of } (\pi|\pi')_{c,c'}\} \leq s_{(\!|\,|\!)}(e_0).
\end{aligned}
$$

So, $s_{(\!|\,|\!)}(e_0) = s(\mathbf{z})-s(\mathcal{W})+s(\mathbf{z}')-s(\mathcal{W}')+2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') = \min\{s_{(\!|\,|\!)}(e); e \text{ is an } (\!|\,|\!)\text{-experiment of } (\pi|\pi')_{c,c'}\}$. Since $\mathcal{W}(e_0) = \mathcal{V}+\mathcal{V}'$ and $\mathit{Card}(\mathcal{V}+\mathcal{V}') = \mathit{Card}(\mathcal{W}+\mathcal{W}')$, we have $s(e_0) = s_{(\!|\,|\!)}(e_0) - 2\,\mathit{Card}(\mathcal{W}+\mathcal{W}') = s(\mathbf{z})-s(\mathcal{W})+s(\mathbf{z}')-s(\mathcal{W}')$.

By Proposition 29, we have $\mathit{length}(R_1) = (s(e_0) - s_{(\!|\,|\!)\,\mathit{inf}}([\![(\pi|\pi')_{c,c'}]\!]))/2 = (s(\mathbf{z})-s(\mathcal{W})+s(\mathbf{z}')-s(\mathcal{W}')-s_{(\!|\,|\!)\,\mathit{inf}}([\![(\pi|\pi')_{c,c'}]\!]))/2$. Moreover, by Lemma 22, there exists $e_1 :_{(\!|\,|\!)} \pi_1$ s.t.

- $\mathcal{W}(e_1) = \mathcal{V}+\mathcal{V}'$
- and $s_{(\!|\,|\!)}(e_1) = \min\{s_{(\!|\,|\!)}(e)\,;\, e \text{ is an } (\!|\,|\!)\text{-experiment of } \pi_1\}$.

By Lemma 25, we have $\mathit{length}(R_2) = \mathit{Card}(\mathcal{V}+\mathcal{V}') = \mathit{Card}(\mathcal{W}+\mathcal{W}')$. Hence

$$
\begin{aligned}
&\quad \mathit{strong}((\pi|\pi')_{c,c'}) \\
&= \mathit{length}(R_1)+\mathit{length}(R_2) \\
&= (s(\mathbf{z})-s(\mathcal{W})+s(\mathbf{z}')-s(\mathcal{W}')+2\,\mathit{Card}(\mathcal{W}+\mathcal{W}')-s_{(\!|\,|\!)\,\mathit{inf}}([\![(\pi|\pi')_{c,c'}]\!]))/2 \\
&= \frac{s(\mathbf{z})+s(\mathcal{W})+2\,\mathit{Card}(\mathcal{W})+s(\mathbf{z}')+s(\mathcal{W}')+2\,\mathit{Card}(\mathcal{W}')-s_{(\!|\,|\!)\,\mathit{inf}}([\![(\pi|\pi')_{c,c'}]\!])}{2} \\
&\quad -s(\mathcal{W}+\mathcal{W}') \\
&= \frac{s_{(\!|\,|\!)}(\mathbf{z},\mathcal{W})+s_{(\!|\,|\!)}(\mathbf{z}',\mathcal{W}')-s_{(\!|\,|\!)\,\mathit{inf}}([\![(\pi|\pi')_{c,c'}]\!])}{2} - s(\mathcal{W}+\mathcal{W}') \\
&= \min(\mathcal{C}).
\end{aligned}
$$

□

In the following example, we apply our results in a concrete case, which also shows that the proof of Theorem 34 *does not* consist in rebuilding $\pi$ and $\pi'$ from

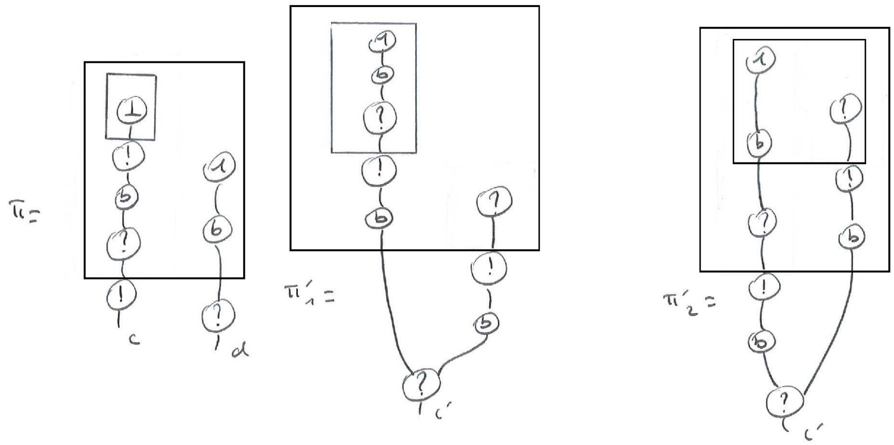

Figure 4: An example

$\llbracket \pi \rrbracket$ and $\llbracket \pi' \rrbracket$ and then compute the number of steps leading from $(\pi|\pi')_{c,c'}$ to a normal form, since in the considered case we know that the point of $\llbracket \pi' \rrbracket$ used to compute $\mathit{strong}((\pi|\pi')_{c,c'})$ is not enough to compute $\pi'$.

**Example 35.** *Let $\pi$ (resp. $\pi'_1$, $\pi'_2$) be the net of Figure 4 with conclusions $d, c$ (resp. $c'$). Notice that we have $\langle(-,[(+,*),(+,*)]),(+,[(-,[(+,[])]),(-,[(+,[(-,*)])])])\rangle \in \llbracket \pi \rrbracket$ and $\langle(-,[(+,[(-,[(+,*)])]),(+,[(-,[])])])\rangle \in \llbracket \pi'_1 \rrbracket, \llbracket \pi'_2 \rrbracket$. We thus have, by Proposition 19,*

- $(\langle(-,[(+,*),(+,*)]),(+,[(-,[(+,[])]),(-,[(+,[(-,*)])])])\rangle,[]) \in \llparenthesis \pi \rrparenthesis$
- *and* $(\langle(-,[(+,[(-,[(+,*)])]),(+,[(-,[])])])\rangle,[]) \in \llparenthesis \pi'_1 \rrparenthesis, \llparenthesis \pi'_2 \rrparenthesis$.

*We have*

- $s_{\llparenthesis\rrparenthesis_{inf}}(\llparenthesis(\pi|\pi'_1)_{c,c'}\rrparenthesis) = s_{\llparenthesis\rrparenthesis_{inf}}(\llparenthesis(\pi|\pi'_2)_{c,c'}\rrparenthesis) = s_{\llparenthesis\rrparenthesis}((\langle(-,[(+,*),(+,*)])\rangle,[])) = 3$,
- $s_{\llparenthesis\rrparenthesis}((\langle(-,[(+,*),(+,*)]),(+,[(-,[(+,[])]),(-,[(+,[(-,*)])])])\rangle,[])) = 9$,
- $s_{\llparenthesis\rrparenthesis}((\langle(-,[(+,[(-,[(+,*)])]),(+,[(-,[])])])\rangle,[])) = 6$,

*hence, by Theorem 34, we have $\mathit{strong}((\pi|\pi'_1)_{c,c'}), \mathit{strong}((\pi|\pi'_2)_{c,c'}) \leq \frac{9+6-3}{2} - 0 = 6$. Actually, these points are those which give the exact value of $\mathit{strong}((\pi|\pi'_1)_{c,c'})$ and of $\mathit{strong}((\pi|\pi'_2)_{c,c'})$: $\mathit{strong}((\pi|\pi'_1)_{c,c'}) = \mathit{strong}((\pi|\pi'_2)_{c,c'}) = 6$, while, as noticed by Pierre Boudes, the points of $\llbracket \pi'_1 \rrbracket$ and of $\llbracket \pi'_2 \rrbracket$ in which the positive multisets are of cardinality $0$ or $1$ are the same.*

## References


[1] Alexis Bernadet and Stéphane Lengrand. Complexity of strongly normalising -terms via non-idempotent intersection types. In Martin Hofmann, editor, *FOSSACS*, volume 6604 of *Lecture Notes in Computer Science*, pages 88–107. Springer, 2011.

[2] M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Principal type schemes and $\lambda$-calculus semantics. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, 1980.

[3] V. Danos and L. Regnier. Proof-nets and the Hilbert space. In *Advances in Linear Logic*, volume 222 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, 1995.

[4] D. de Carvalho. *Sémantiques de la logique linéaire et temps de calcul*. Thèse de doctorat, Université Aix-Marseille II, 2007.

[5] D. de Carvalho. Execution time of lambda-terms via denotational semantics and intersection types. RR 6638, INRIA, 2008. To appear in Mathematical Structures in Computer Science.

[6] D. de Carvalho, M. Pagani, and L. Tortora de Falco. A semantic measure of the execution time in linear logic. *Theor. Comput. Sci.*, 412/20:1884–1902, 2011.

[7] Thomas Ehrhard and Laurent Regnier. Böhm trees, Krivine machine and the Taylor expansion of ordinary lambda-terms. In *Approaches to Computational Barriers. Second Conference on Computability in Europe, CiE 2006, Swansea, UK*, Lecture Notes in Computer Science. Springer, 2006.

[8] J.-Y. Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.

[9] Michele Pagani and Lorenzo Tortora de Falco. Strong normalization property for second order linear logic. *Theoretical Computer Science*, 441/2:410–444, 2010.

# Proof of Proposition 6 and Proposition 7

**Lemma 36.** *Assume that $\pi \rightsquigarrow_e \pi_1$ and $\pi_1 \rightsquigarrow_{\neg e} \pi''$. Then there exist $\pi'$ such that $\pi \rightsquigarrow_{\neg e} \pi'$ and a non-empty sequence reduction $\pi' \rightsquigarrow^* \pi''$:*

$$\begin{tikzcd}
\pi \arrow[r, "e"] \arrow[d, dashed, "\neg e"'] & \pi_1 \arrow[d, "\neg e"] \\
\pi' \arrow[r, dashed, two heads, "\geq 1"] & \pi''
\end{tikzcd}$$

**Proposition 37** (postponing erasing steps)**.** *For any net $\pi_0$ such that there is no infinite reduction sequence from $\pi_0$, for any finite reduction sequence $R$ from $\pi_0$ to $\pi'$, there exist a $\neg e$-reduction sequence $R'$ from $\pi_0$ to some net $\pi$ and an $e$-reduction sequence $R_0$ from $\pi$ to $\pi'$ such that $length(R) \leq length(R') + length(R_0)$.*

*Proof.* By induction on $\max\{length(R); R \text{ is a reduction sequence from } \pi_0\}$. Let $R$ be a finite reduction sequence $\pi_0 \rightsquigarrow \pi_1 \rightsquigarrow \dots \pi_{n-1} \rightsquigarrow \pi_n = \pi'$. If $R$ has no $\neg e$-reduction steps, then we set $\pi = \pi_0$ and $R_0 = R$. Otherwise, we set $k = \min\{i \in \mathbb{N}; \pi_i \rightsquigarrow_e \pi_{i+1}\}$: if $k > 0$, then we apply the induction hypothesis to $\pi_1$; if $k = 0$, then we set $r = \min\{j \in \mathbb{N}; \pi_j \rightsquigarrow_{\neg e} \pi_{j+1}\}$; we apply $r$ times Lemma 36, we thus obtain a reduction sequence $R_1$ from $\pi_0$ to $\pi_{r+1}$ of length $\geq r+1$ in which the first reduction step $\pi_0 \rightsquigarrow \pi'_1$ is non-erasing. We can thus consider the reduction sequence $R_1$ followed by the reduction sequence $\pi_{r+1} \rightsquigarrow \pi_{r+2} \rightsquigarrow \dots \pi_{n-1} \rightsquigarrow \pi_n$ and apply the induction hypothesis to $\pi'_1$. □

**Lemma 38.** *Assume that $\pi \rightsquigarrow_{\neg e} \pi_1$ and $\pi \rightsquigarrow_{(\neg e)_s} \pi'$ with $\pi' \neq \pi_1$. Then there exist $\pi''$ such that $\pi_1 \rightsquigarrow_{(\neg e)_s} \pi''$ and a non-empty reduction sequence $\pi' \rightsquigarrow_{\neg e}{}^* \pi''$:*

$$\begin{tikzcd}
\pi \arrow[r, "\neg e"] \arrow[d, "(\neg e)_s"'] & \pi_1 \arrow[d, dashed, "(\neg e)_s"] \\
\pi' \arrow[r, dashed, two heads, "\geq 1", "\neg e"'] & \pi''
\end{tikzcd}$$

*Proof.* Let $x$ (resp. $y$) be the cut-link reduced by the step $\pi \rightsquigarrow_{\neg e} \pi_1$ (resp. $\pi \rightsquigarrow_{(\neg e)_s} \pi'$): we know by hypothesis that $x \neq y$. Since $x$ is non erasing and $y$ is stratified, there exists a unique residue $\vec{y}$ of $y$ in $\pi_1$. Since $y$ is non erasing and $x$ needs not being stratified, there exist $n \geq 1$ residues $\vec{x_1}, \dots, \vec{x_n}$ of $x$ in $\pi'$. The net $\pi''$ can be obtained both by reducing $\vec{y}$ in $\pi_1$ and by reducing $\vec{x_1}, \dots, \vec{x_n}$ in $\pi'$. □

**Lemma 39.** *For any net $\pi_0$ such that*

- *there is no infinite* $\neg e$*-reduction sequence from* $\pi_0$
- *and there is some net* $\pi'$ *with some clash such that* $\pi_0 \leadsto_{\neg e}{}^* \pi'$*,*

*for any net* $\pi'_0$ *such that* $\pi_0 \leadsto_{(\neg e)_s} \pi'_0$*, there exists a net* $\pi''$ *with some clash such that* $\pi'_0 \leadsto^*_{(\neg e)_s} \pi''$*.*

*Proof.* By induction on $\max\{\mathit{length}(R); R$ is a $\neg e$-reduction sequence from $\pi_0\}$. We consider a $\neg e$-reduction sequence $\pi_0 \leadsto_{\neg e} \pi_1 \leadsto_{\neg e} \ldots \pi_n \leadsto_{\neg e} \pi_{n+1}$, where $\pi_{n+1}$ is a net with some clash. Let $\pi'_0$ be a net such that $\pi_0 \leadsto_{(\neg e)_s} \pi'_0$.

- If $\pi'_0 = \pi_1$ and $n = 0$, then we set $\pi'' = \pi_1$.
- If $\pi'_0 = \pi_1$ and $n \neq 0$, then we apply the induction hypothesis to $\pi_1$.
- If $\pi'_0 \neq \pi_1$, then, by Lemma 36, there exists a net $\pi''_0$ such that $\pi'_0 \leadsto_{\neg e}{}^* \pi''_0$ and $\pi_1 \leadsto_{(\neg e)_s} \pi''_0$; by induction hypothesis on $\pi_1$, there exists a net $\pi''_1$ with some clash such that $\pi''_0 \leadsto^*_{(\neg e)_s} \pi''_1$; finally we apply the induction hypothesis on $\pi'_0$.

□

**Proposition 40.** *We have* $\boldsymbol{SN}^{\neg e} = \boldsymbol{SN}^{(\neg e)_s}$*.*

*Proof.* The inclusion $\mathbf{SN}^{\neg e} \subseteq \mathbf{SN}^{(\neg e)_s}$ is trivial.

If $\pi \notin \mathbf{SN}^{\neg e}$, then we are in one the two following cases:

1. - there is no infinite $\neg e$-reduction sequence from $\pi$
   - and there is some net $\pi'$ with some clash such that $\pi \leadsto_{\neg e}{}^* \pi'$,
2. or there exists an infinite $\neg e$-reduction sequence from $\pi$.

In the first case, we just apply Lemma 39.

Now, we prove that, for any net $\pi$ from which there exists an infinite $\neg e$-reduction sequence, for any net $\pi'$ such that $\pi \leadsto_{(\neg e)_s} \pi'$, there exists an infinite $\neg e$-reduction sequence from $\pi'$: this is enough to show that there exists an infinite $(\neg e)_s$-reduction sequence from $\pi$. Indeed, for any $\pi$ from which there exists an infinite $\neg e$-reduction sequence, there always exists $\pi'$ such that $\pi \leadsto_{(\neg e)_s} \pi'$, and thus from an infinite $\neg e$-reduction sequence starting from $\pi$ one can build an infinite $(\neg e)_s$-reduction sequence strating from $\pi$.
More precisely, we prove, by induction on $n$, that, for any $n \in \mathbb{N}$, for any net $\pi$ from which there exists an infinite $\neg e$-reduction sequence, for any net $\pi'$ such that $\pi \leadsto_{(\neg e)_s} \pi'$, there exists a sequence of $\neg e$-reduction steps $\pi' = \pi''_0 \leadsto_{\neg e} \pi''_1 \leadsto_{\neg e} \ldots \pi''_{n-1} \leadsto_{\neg e} \pi''_n$.

Let $\pi = \pi_0 \leadsto_{\neg e} \pi_1 \leadsto_{\neg e} \ldots \pi_i \leadsto_{\neg e} \pi_{i+1} \ldots$ be an infinite sequence of $\neg e$-reduction steps and assume that $\pi \leadsto_{(\neg e)_s} \pi'$. If $\pi' = \pi_1$, then we are done. Otherwise, by Lemma 38, there exist $\pi''$ such that $\pi_1 \leadsto_{(\neg e)_s} \pi''$ and a non-empty $\neg e$-reduction sequence $\pi' \leadsto_{\neg e}{}^* \pi''$; since there exists an infinite $\neg e$-reduction sequence from $\pi_1$ and $\pi_1 \leadsto_{(\neg e)_s} \pi''$, we can apply the induction hypothesis: there exists a sequence of $\neg e$-reduction steps $\pi'' = \pi''_0 \leadsto_{\neg e} \pi''_1 \leadsto_{\neg e} \ldots \pi''_{n-1} \leadsto_{\neg e} \pi''_n$, hence there exists a sequence of (at least) $n + 1$ $\neg e$-reduction steps from $\pi'$. □

**Fact 41.** *If $\pi \rightsquigarrow_e{}^* \pi'$ and $\pi'$ contains some clash, then the net $\pi$ contains some clash too.*

**Proposition 6.** *We have $\boldsymbol{SN} = \boldsymbol{SN}^{(\neg e)_s}$.*

*Proof.* By Proposition 40, it is sufficient to show that $\mathbf{SN} = \mathbf{SN}^{\neg e}$.
If $\pi \notin \mathbf{SN}$, then we are in one the two following cases:

1. • there is no infinite reduction sequence from $\pi$
   • and there is some net $\pi'$ with some clash such that $\pi \rightsquigarrow^* \pi'$,
2. or there exists an infinite reduction sequence from $\pi$.

Assume that we are in the first case. Then, by Proposition 37, there exist a $\neg e$-reduction sequence $R$ from $\pi$ to $\pi_1$ and an $e$-reduction sequence from $\pi_1$ to $\pi'$, where $\pi'$ is a net containing some clash. By Fact 41, the net $\pi_1$ contains some clash too, hence $\pi \notin \mathbf{SN}^{\neg e}$.

Now, if we are in the second case, we have to show that there exists an infinite $\neg e$-reduction sequence from $\pi$. This has been showed in [9] using Lemma 36. $\square$

**Proposition 42.** *For any $\pi_0 \in \boldsymbol{SN}^{\neg e}$, for any $\neg e$-reduction sequence $R'''$ from $\pi_0$ to a $\neg e$-normal form $\pi$, there exists a $(\neg e)_s$-reduction sequence $R_1$ from $\pi_0$ to $\pi$ such that $length(R''') \leq length(R_1)$.*

*Proof.* We prove, by induction on $strong_{\neg e}(\pi_0)$, that, for any $\pi_0 \in \mathbf{SN}^{\neg e}$, for any $\neg e$-reduction sequence $R'''$ from $\pi_0$ to a $\neg e$-normal form $\pi$, for any $\pi'$ such that $\pi \rightsquigarrow_{(\neg e)_s} \pi'$, there exists a $(\neg e)_s$-reduction sequence $R_1$ from $\pi'$ to $\pi$ such that $length(R''') \leq length(R_1) + 1$.

- If $strong_{\neg e}(\pi_0) = 0$, then there is no such $\pi'$.
- If $strong_{\neg e}(\pi_0) > 0$, then we apply Lemma 38 and the induction hypothesis. More precisely, suppose that $R'''$ is such that $\pi_0 \rightsquigarrow_{\neg e} \pi_1 \rightsquigarrow_{\neg e}{}^* \pi$. If $\pi_0 \rightsquigarrow_{(\neg e)_s} \pi_1$, then we apply the induction hypothesis to $\pi_1$. Otherwise, there exists $\pi' \neq \pi_1$ such that $\pi_0 \rightsquigarrow_{(\neg e)_s} \pi'$ and we can apply Lemma 38: there exist $\pi''$ such that $\pi_1 \rightsquigarrow_{(\neg e)_s} \pi''$ and a non-empty reduction sequence $\pi' \rightsquigarrow_{\neg e}{}^* \pi''$. We can call $R_1'''$ the $\neg e$-reduction sequence leading from $\pi_1$ to $\pi$ and apply the induction hypothesis to $\pi_1$: there exists a $\neg e$-reduction sequence $R_1^1$ from $\pi''$ to $\pi$ such that $length(R_1''') \leq length(R_1^1) + 1$. Now, since there exists a non-empty reduction sequence $\pi' \rightsquigarrow_{\neg e}{}^* \pi''$, there also exists a $\neg e$-reduction sequence $R_2'''$ from $\pi'$ to $\pi$ such that $length(R_2''') \geq length(R_1^1) + 1$. By applying the induction hypothesis to $\pi'$, there exists a $(\neg e)_s$-reduction sequence $R_3'''$ from $\pi'$ to $\pi$ such that $length(R_3''') \geq length(R_2''')$. We consider $R_1$ defined by $\pi_0 \rightsquigarrow_{(\neg e)_s} \pi'$ followed by $R_3'''$. We have $length(R_1) = length(R_3''') + 1 \geq length(R_2''') + 1 \geq length(R_1^1) + 1 + 1 \geq length(R_1''') + 1 = length(R''')$.

$\square$

**Fact 43.** *If $\pi \rightsquigarrow_{\neg e} \pi'$, then $\pi'$ has at least $n-1$ cut-links, where $n$ is the number of cut-links in $\pi$.*

**Lemma 44.** *Let $\pi_0 \in \boldsymbol{SN}$ with at least $n$ cut-links. Then there exist*

- *a $\neg e$-normal net $\pi$;*
- *a $\neg e$-reduction sequence $R_1$ from $\pi_0$ to $\pi$;*
- *and an antistratified $e$-reduction sequence $R_2$ from $\pi$*

*such that $n \leq \mathit{length}(R_1) + \mathit{length}(R_2)$.*

*Proof.* By induction on $\mathit{strong}(\pi_0)$. We distinguish between two cases:

- There exists $\pi_1$ such that $\pi_0 \rightsquigarrow_{\neg e} \pi_1$: we apply Fact 43 and the induction hypothesis on $\pi_1$.
- The net $\pi_0$ is $\neg e$-normal: we take for $R_1$ the empty reduction sequence from $\pi_0$ to $\pi_0$ and for $R_2$ an antistratified $e$-reduction sequence $\pi_0 \rightsquigarrow_e \pi_1 \ldots \rightsquigarrow_e \pi_n$ such that, for any $i \in \{0, \ldots, n\}$, the net $\pi_i$ has exactly $n-i$ erasing cut-links.

□

**Fact 45.** *Let $R_0$ be an $e$-reduction sequence from $\pi'$. Then $\pi'$ has at least $\mathit{length}(R_0)$ cut-links.*

**Proposition 7.** *For any $\pi_0 \in \boldsymbol{SN}$, there exist a $(\neg e)_s$-reduction sequence $R_1 : \pi_0 \rightsquigarrow^*_{(\neg e)_s} \pi$ with $\pi$ $\neg e$-normal and an antistratified $e$-reduction sequence $R_2$ from $\pi$ such that $\mathit{strong}(\pi_0) = \mathit{length}(R_1) + \mathit{length}(R_2)$.*

*Proof.* Let $\pi_0 \in \mathbf{SN}$ and let $R$ be a reduction sequence from $\pi_0$. By Proposition 37, there exist a $\neg e$-reduction sequence $R'$ from $\pi_0$ to some net $\pi'$ and an $e$-reduction sequence $R_0$ from $\pi'$ such that $\mathit{length}(R) \leq \mathit{length}(R') + \mathit{length}(R_0)$. By Fact 45, the net $\pi'$ has at least $\mathit{length}(R_0)$ cut-links, hence, by Lemma 44, there exist

- a $\neg e$-normal net $\pi$;
- a $\neg e$-reduction sequence $R''$ from $\pi'$ to $\pi$;
- and an antistratified $e$-reduction sequence $R_2$ from $\pi$

such that $\mathit{length}(R_0) \leq \mathit{length}(R'') + \mathit{length}(R_2)$. We consider $R'''$ defined by $R'$ followed by $R''$. By Proposition 42, there exists a $(\neg e)_s$-reduction sequence $R_1$ from $\pi_0$ to $\pi$ such that $\mathit{length}(R_1) \geq \mathit{length}(R''')$. We thus have: $\mathit{length}(R_1) + \mathit{length}(R_2) \geq \mathit{length}(R') + \mathit{length}(R'') + \mathit{length}(R_2) \geq \mathit{length}(R') + \mathit{length}(R_0) \geq \mathit{length}(R)$. By taking as $R$ any reduction sequence such that $\mathit{length}(R) = \mathit{strong}(\pi_0)$, we obtain the required $R_1$ and $R_2$. □